NAME:

Solutions to Math 150 Practice Exam 1.1

Instructions: WRITE YOUR NAME CLEARLY. Do as many problems as you can for a maximal score of 100. SHOW YOUR WORK!

1. Given that $\lim_{x\to 1} f(x) = 8$, $\lim_{x\to 1} g(x) = 3$, and $\lim_{x\to 1} h(x) = 2$ find

a) $\lim_{x\to 1} \frac{f(x)}{g(x)-h(x)}$ [5 pts]

Solution: $\lim_{x\to 1} \frac{f(x)}{g(x)-h(x)} = \frac{8}{3-2} = 8$

b) $\lim_{x\to 1} \sqrt[3]{f(x)g(x)+3}$ [5 pts]

Solution: $\lim_{x\to 1} \sqrt[3]{f(x)g(x)+3} = \sqrt[3]{8\cdot 3+3} = \sqrt[3]{27} = 3$

2. Use the squeeze theorem to evaluate $\lim_{x\to 0^+} \sqrt{x}\ Sin\left(\frac{\pi}{x}\right)$ [10 pts]

Solution: $-1 \le Sin\left(\frac{\pi}{x}\right) \le 1$. Therefore for any $x \ge 0$,

$$-\sqrt{x} \le \sqrt{x} Sin\left(\frac{\pi}{x}\right) \le \sqrt{x}.$$

Since $\lim_{x\to 0^+} \sqrt{x} = \lim_{x\to 0^+} -\sqrt{x} = 0$, it follows by the squeeze theorem that $\lim_{x\to 0^+} \sqrt{x} Sin\left(\frac{\pi}{x}\right) = 0$.

3. Evaluate $\lim_{x \to -\infty} \frac{\sqrt{16x^4 + 64x^2} + x^2}{2x^2 - 4}$ [10 pts]

Solution: $\lim_{x\to-\infty} \frac{\sqrt{16x^4+64x^2}+x^2}{2x^2-4} = \lim_{x\to-\infty} \frac{x^2\left(\sqrt{16+\frac{64}{x^2}}+1\right)}{x^2\left(2-\frac{4}{x^2}\right)} = \frac{5}{2}$

4. Find an equation of the tangent line to the curve $y = 4x^2 + 2x$ at the point $a = -2$. [10 pts]

Solution: $\boldsymbol{\frac{dy}{dx}} = 8x + 2$. Therefore $\frac{dy}{dx}|_{x=-2} = 8(-2) + 2 = -14$. The tangent line must pass through the point (-2, 12) so the equation is

$$y - 12 = -14(x + 2)$$

The equation may also be written as

$$y = -14x - 16$$

5. Find the derivative of the function $f(x)=\sqrt{x+2}$ using the definition of the derivative at the point $a=7$. [10 pts]

Solution: $\lim_{x\to 7}\frac{\sqrt{x+2}-\sqrt{7+2}}{(x-7)}=\lim_{x\to 7}\frac{(\sqrt{x+2}-\sqrt{9})(\sqrt{x+2}+\sqrt{9})}{(x-7)(\sqrt{x+2}+\sqrt{9})}$

Thus, $\lim_{x\to 7}\frac{(x+2)-9}{(x-7)(\sqrt{x+2}+\sqrt{9})}=\lim_{x\to 7}\frac{1}{(\sqrt{x+2}+\sqrt{9})}=\frac{1}{6}$

6. Evaluate $\lim\limits_{x\to 2}\frac{x^5-32}{x-2}$ [10 pts]

Solution: $\lim_{x\to 2}\frac{x^5-32}{x-2}=\lim_{x\to 2}\frac{(x-2)(x^4+2x^3+4x^2+8x+16)}{(x-2)}=5\cdot 16$

7. Let $f:[0, 1] \to (0, 1)$ be a continuous function such that $0 < f(x) < 1$ for all $x \in [0, 1]$. Prove that the equation $f(x) = x$ has a solution for at least one $x \in [0, 1]$.

[10 pts]

Solution: Define $g(x) = f(x) - x$. Then $g(x)$ is continuous. Furthermore

(i) $g(0) = f(0) - 0 = f(0) > 0$

(ii) $g(1) = f(1) - 1 < 1 - 1 = 0$

Hence, by the Intermediate-Value theorem, the equation $g(x) = 0$ must have a solution for at least one x. This, in turn, implies that a solution to the equation $f(x) = x$ must exist.

8. Let $a > 0$ be a positive real number. Define $f(x) = \begin{cases} x & if \quad x < a \\ 3x - 2 & if \quad x \geq a \end{cases}$.

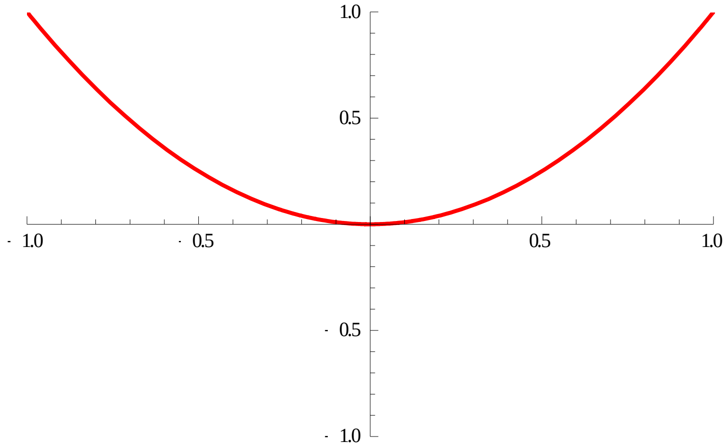
What is the value of a if f is continuous on the entire real number line? [10 pts]

Solution: In order for the function $f(x)$ to be continuous, we must have

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a).$$

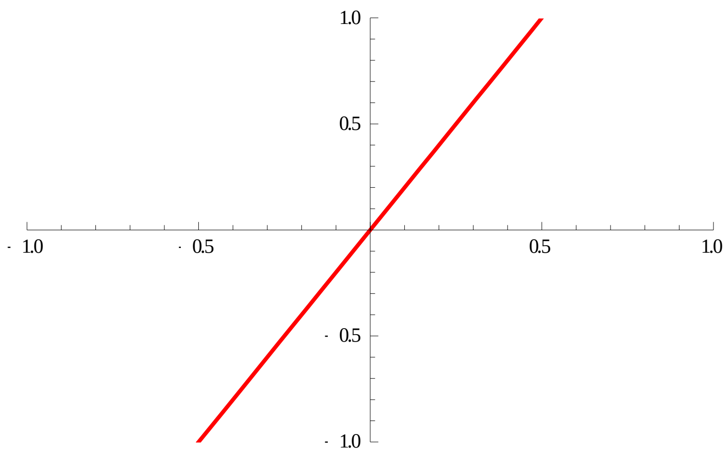
Thus, $a = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = 3a - 2$. Hence $a = 1$.

9. The graph of the function $y = f(x)$ is displayed below



Draw the graph of $y = f\,'(x)$. [10 pts]

Solution:



10. Evaluate $\lim_{x \to 0} \frac{\sin 3x}{x}$ [10 pts]

Solution: We know that $\lim_{h\to 0} \frac{\text{Sin}\, h}{h} = 1.$ Therefore
$\lim_{x\to 0} \frac{Sin3x}{x} = \lim_{x\to 0} 3\frac{Sin3x}{3x} = 3$

Extra-Credit

11. Prove by means of a delta-epsilon argument that $\lim_{x\to 2}(3x-1)=5$

[10 pts]

Solution: $|3x-1-5| = |3x-6| = 3|x-2| < \epsilon$.
Therefore set $\delta(\epsilon) = \epsilon/3$.

12. Establish the derivative product formula. Namely, show that $(fg)' = f'g + fg'$

[10 pts]

Solution: $(fg)'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} =$
$\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} =$
$\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h} + \frac{f(x)g(x+h)-f(x)g(x)}{h} =$
$\lim_{h\to 0} g(x+h)\frac{f(x+h)-f(x)}{h} + f(x)\frac{g(x+h)-g(x)}{h} = f'(x)g(x) + f(x)g'(x)$.